

The unitary transformation of the path-integral measure

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Abstract

The aim of the article is to show how a coordinate transformation can be applied to the path-integral formalism. For this purpose the unitary definition of the quantum measure, which guarantees the conservation of total probability, is offered. As the examples, the phase space transformation to the canonically conjugate pair (*energy, time*) and the transformation to the cylindrical coordinates are shown. The transformations of the path-integral measure looks classically but they can not be deduced from naive transformations of quantum trajectories.

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1 Introduction

There is the remarkable fact that the integrable quantum systems are quasiclassical in many aspects. For instance, the energy spectra of the H -atom problem [1], of the Pöschle-Teller problem [2], of the rigid rotator problem [3] and even of the *sine*-Gordon problem [4] are quasiclassical. This fact should have the explanation. The Dowker's theorem [5] which insist that the quasiclassical approximation is exact for path integrals on the simple Lie group manifolds formally explains this phenomena. So, one can expect that in curved spaces (where the classical arguments play the crucial role) the quantum-mechanical problems are solvable, or at least become transparent.

However we know how to construct correctly the path integral formalism only in the flat space [6]. Then in the curved space the path integrals can be defined through the corresponding coordinate transformations. But there is the opinion that it is impossible to perform the transformation of path-integral variables: the naive coordinate transformation give wrong result since the stochastic nature of quantum trajectories. One can find the examples in [6, 7]. We intend here to consider this problem.

The mostly powerful method of coordinate transformations in the path-integral formalism is the "time-sliced" method [8]. In frame of this method a number of quantum problems were solved [9]. But it is too cumbersome for analytical manipulations and, moreover, it leads to unwanted time-slicing corrections (see also [10]).

The stochastic nature of quantum trajectories suggests an idea that it is possible to loss, or to add, contributions uncontrollably when the transformations are performed. That is why such general principle as the conservation of total probability should play important role. The purpose of this article is to show how the S -matrix unitarity condition can be adopted in the path- integral formalism to solve the problem of coordinate transformations (preliminaries were given in [11]).

The unitarity condition for the S -matrix

$$SS^+ = S^+S = 1 \quad (1.1)$$

presents the infinite set of nonlinear equalities for elements of the S -matrix:

$$iAA^* = A - A^*, \quad (1.2)$$

where A is the amplitude, $S = 1 + iA$. Expressing the amplitude through the path integral one can see that the left hand side of (1.2) offers the double integral and, at the same time, the right hand side is the linear combination of single integrals. Let us consider what it give to us.

Using the spectral representation of one-particle amplitude:

$$A(x_1, x_2; E) = \sum_n \Psi_n(x_2) \frac{1}{E - E_n + i\varepsilon} \Psi_n^*(x_1), \quad \varepsilon \rightarrow +0, \quad (1.3)$$

let us calculate

$$R(E) = \int dx_1 dx_2 A(x_1, x_2; E) A^*(x_1, x_2; E). \quad (1.4)$$

The integration over end points x_1 and x_2 is performed for the sake of simplicity only. Since the orthonormalizability of the wave functions $\Psi_n(x)$ we will find that

$$R(E) = \sum_n \left| \frac{1}{E - E_n + i\varepsilon} \right|^2 = \sum_n \frac{i}{2\varepsilon} \left(\frac{1}{E - E_n + i\varepsilon} - \frac{1}{E - E_n - i\varepsilon} \right) = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n), \quad (1.5)$$

It is well known that representation (1.3) satisfies the unitarity condition (1.2). But it is important for us that $R(E) = 0$ for all $E \neq E_n$. It is evident that all unnecessary contributions with $E \neq E_n$ were canceled by difference in the right hand side of eq.(1.2). We will put this phenomena in the basis of our approach.

To see the integral form of this cancelation phenomena let us use the proper-time representation:

$$A(x_1, x_2; E) = \sum_n \Psi_n(x_1) \Psi_n^*(x_2) i \int_0^\infty dT E^{i(E - E_n + i\varepsilon)T} \quad (1.6)$$

and insert it into (1.4):

$$R(E) = \sum_n \int_0^\infty dT_+ dT_- e^{-(T_+ + T_-)\varepsilon} e^{i(E - E_n)(T_+ - T_-)}. \quad (1.7)$$

We will introduce new time variables instead of T_\pm :

$$T_\pm = T \pm \tau, \quad (1.8)$$

where, it follows from Jacobian of transformation, $|\tau| \leq T$, $0 \leq T \leq \infty$. But we can put $|\tau| \leq \infty$ since $T \sim 1/\varepsilon \rightarrow \infty$ is essential in (1.7). In result,

$$R(E) = 2\pi \sum_n \int_0^\infty dT e^{-2\varepsilon T} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi} e^{2i(E - E_n)\tau}. \quad (1.9)$$

In the last integral all contributions with $E \neq E_n$ are really canceled and

$$R(E) = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n). \quad (1.10)$$

Note that the product of amplitudes AA^* was “linearised” after introduction of “virtual” time $\tau = (T_+ - T_-)/2$. The physical meaning of such variables was discussed firstly in [12] (see also [11] and Sec.3).

We will see that this cancelation mechanism unambiguously determines the path-integral measure and, in result, it allows to perform the transformation to arbitrary useful variables. So, in this article we will build the perturbation theory in a curved space for $R(E)$ omitting calculation of amplitudes. It leads to losses of some information since the amplitudes can be restored in this approach with a phase accuracy only. But it is sufficient for calculation of the energy spectrum.

In this paper the following statements will be demonstrated.

1. The unitarity condition unambiguously determines contributions in the path integrals.

This statement looks like a tautology since $\exp\{iS(x)\}$, where $S(x)$ is the action, is the unitary operator which shifts the system along the trajectory x . I.e. the unitarity is already fixed in the path-integral formalism. But the above considered cancellation of the real part of the amplitude requires that the general path-integral solution contains the unnecessary (i.e. unobservable) degrees of freedom. This means that (1.1) is the necessary condition. We want to show that it is the sufficient also, unambiguously determines the quantum trajectories.

We start the consideration from simplest “0-dimensional” x^3 model to demonstrate quantitatively the main ideas and technical tricks (Sec.2) and will extend the formalism on quantum mechanics in Sec.3. The secondary result of our approach is the demonstration of the fact that the stationary phase method of calculation of path integrals conserves the total probability.

2. The description of quantum-mechanical perturbations may be reduced by the canonical transformations to the counting of local fluctuations, with known weight function, of group manifold.

The proof of this statement will be given in Sec.4. It based on the unitary definition of quantum measure. The perturbation theory becomes free from the doubling of degrees of freedom in spite of the double path integrals are calculated.

3. The Jacobian of transformations can be reduced to one by the suitable chosen quantum measure, i.e. of the weight function.

We will show this in Sec.5 considering noncanonical transformation to the cylindrical coordinates. This statement noticeably simplify the calculations.

It must be noted that these results can not be deduced by the naive transformations of path integral variables.

It will be shown that even on the simple Lee group manifolds the quasiclassical approximation is not free from quantum corrections in our approach, since we start the calculations from path integrals defined in the Cartesian coordinates. The connection with the Dowker’s theorem [5] will be discussed briefly in Sec.6.

2 0-dimensional model

Let us consider the integral:

$$A = \int_{-\infty}^{+\infty} \frac{dx}{(2\pi)^{1/2}} e^{i(\frac{1}{2}ax^2 + \frac{1}{3}bx^3)}, \quad (2.1)$$

with $Ima \rightarrow +0$ and $b > 0$. We want to compute the “probability”

$$R = |A|^2 = \int_{-\infty}^{+\infty} \frac{dx_+ dx_-}{2\pi} e^{i(\frac{1}{2}ax_+^2 + \frac{1}{3}bx_+^3) - i(\frac{1}{2}ax_-^2 + \frac{1}{3}bx_-^3)}. \quad (2.2)$$

As in (1.7) we will introduce new variables:

$$x_{\pm} = x \pm e. \quad (2.3)$$

In result:

$$R = \int_{-\infty}^{+\infty} \frac{dxde}{\pi} e^{-2(x^2+e^2)Ima} e^{i(Rea x + 2bx^2)e} e^{2i\frac{b}{3}e^3}. \quad (2.4)$$

Note that integration is performed along the real axis for simplicity.

We will compute the integral over e perturbatively. For this purpose the transformation:

$$F(e) = e^{\frac{1}{2i}\hat{j}\hat{e}'} e^{2ije} F(e'), \quad (2.5)$$

which is valid for any differentiable function, is useful. In (2.5) two auxiliary variables j and e' were introduced and the “hat” symbol means the differentiation over corresponding quantity:

$$\hat{j} = \frac{\partial}{\partial j}, \quad \hat{e}' = \frac{\partial}{\partial e'}. \quad (2.6)$$

At the end of calculations the auxiliary variables must be taken equal to zero.

Choosing

$$\ln F(e) = -2e^2 Ima + 2i\frac{b}{3}e^3 \quad (2.7)$$

we will find:

$$R = e^{\frac{1}{2i}\hat{j}\hat{e}} \int_{-\infty}^{+\infty} dx e^{-2(x^2+e^2)Ima} e^{2i\frac{b}{3}e^3} \delta(Rea x + bx^2 + j). \quad (2.8)$$

Therefore, the destructive interference among two exponents in product AA^* unambiguously determines both integrals, over x and over e . The integral over difference $e = (x_+ - x_-)/2$ gives δ -function and then this δ -function defines the contributions in the last integral over $x = (x_+ + x_-)/2$. Since δ -function is the “zero-width function” only strict solutions of equation

$$Rea x + bx^2 + j = 0 \quad (2.9)$$

give the contribution in R . It is the reason why the discussed cancelation mechanism will determine unambiguously the quantum measure.

But one can note that this is not the complete solution of the problem: the expansion of operator exponent $\exp\{\frac{1}{2i}\hat{j}\hat{e}\}$ generates the asymptotic series. Note also that it is impossible to remove the source j dependence (only harmonic case $b = 0$ is free from j).

The exponent in (2.1) has two extremums at $x_1 = 0$ and at $x_2 = -a/b$. Performing trivial transformation $e \rightarrow ie$, $\hat{e} \rightarrow -i\hat{e}$ of auxiliary variable we find at the limit $Ima = 0$ that the contribution from x_1 extremum (minimum) gives expression:

$$R = \frac{1}{a} e^{-\frac{1}{2}\hat{j}\hat{e}} (1 - 4bj/a^2)^{-1/2} e^{2\frac{b}{3}e^3} \quad (2.10)$$

and the expansion of operator exponent gives the asymptotic series:

$$R = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{(6n-1)!!}{n!} \left(\frac{2b^4}{3a^6}\right)^n, \quad (-1)!! = 0!! = 1. \quad (2.11)$$

This series is convergent in Borel’s sense [13].

Let us calculate now R using stationary phase method. The contribution from the minimum x_1 gives ($Ima = 0$):

$$A = e^{-i\hat{j}\hat{x}} e^{-\frac{i}{2a}j^2} e^{i\frac{b}{3}x^3} \left(\frac{i}{a}\right)^{1/2}. \quad (2.12)$$

The corresponding “probability” is

$$R = \frac{1}{a} e^{-i(\hat{j}_+\hat{x}_+ - \hat{j}_-\hat{x}_-)} e^{-\frac{i}{2a}(j_+^2 - j_-^2)} e^{i\frac{b}{3}(x_+^3 - x_-^3)}. \quad (2.13)$$

Introducing the new auxiliary variables:

$$j_{\pm} = j \pm j_1, \quad x_{\pm} = x \pm e \quad (2.14)$$

and, correspondingly,

$$\hat{j}_{\pm} = (\hat{j} \pm \hat{j}_1)/2, \quad \hat{x}_{\pm} = (\hat{x} \pm \hat{e})/2 \quad (2.15)$$

we find from (2.13):

$$R = \frac{1}{a} e^{-\frac{1}{2}\hat{j}\hat{e}} e^{2\frac{b}{3}e^3} e^{\frac{2b}{a^2}ej^2} \quad (2.16)$$

This expression does not coincide with (2.10) but it leads to the same asymptotic series (2.11). We may conclude that both considered methods of calculation of R are equivalent since the Borel’s regularization scheme of asymptotic series gives the unique result.

The difference between this two methods of calculation is in different organization of perturbations. So, if $F(e)$, instead of (2.7), is chosen in the form:

$$\ln F(e) = -2e^2 Ima + 2i\frac{b}{3}e^3 + 2ibx^2e, \quad (2.17)$$

we may find (2.16) straightforwardly. Therefore, our method has the freedom in choice of (quantum) source j .

The transition from perturbation theory with eq.(2.7) to the theory with eq.(2.17) formally looks like the following transformation of δ -function:

$$\delta(ax + bx^2 + j) = e^{-i\hat{j}'\hat{e}'} e^{i(bx^2 + j)e'} \delta(ax + j'). \quad (2.18)$$

Here the transformation (2.5) was used. Inserting eq.(2.18) into (2.8) we easily find (2.16). Performing the coordinate transformations it is useful for analytic calculations to have quantum sources which correspond to the new variables. Formally this transition is equivalent to the transformation (2.18). Note that this transformation will not lead to changing of the Borel’s regularization procedure.

3 The unitary definition of path integral measure

Let us consider the one dimensional motion. The corresponding amplitude has the form:

$$A(x_1, x_2; E) = i \int_0^\infty dT e^{iET} \int D_{C_+} x e^{iS_{C_+}(x)} \delta(x_1 - x(0)) \delta(x_2 - x(T)), \quad (3.1)$$

where the action

$$S_{C_+}(x) = \int_{C_+} dt \left(\frac{1}{2} \dot{x}^2 - v(x) \right) \quad (3.2)$$

and the measure

$$D_{C_+} x = \prod_{t \in C_+} \frac{dx(t)}{(2\pi)^{1/2}} \quad (3.3)$$

are defined on the shifted in the upper half plane Mills' time contour $C_+ = C_+(T)$ [14]:

$$t \rightarrow t + i\varepsilon, \quad \varepsilon > 0, \quad 0 \leq t \leq T. \quad (3.4)$$

Such definition of the time contour guaranties the convergence of path integral.

Inserting (3.1) into (1.4) we find:

$$R(E) = \int_0^\infty e^{iE(T_+ - T_-)} \int D_{C_+} x_+ D_{C_-} x_- \times \\ \times \delta(x_+(0) - x_-(0)) \delta(x_+(T_+) - x_-(T_-)) e^{iS_{C_+(T_+)}(x_+) - iS_{C_-(T_-)}(x_-)}, \quad (3.5)$$

where $C_-(T) = C_+^*(T)$ is the time contour in the lower complex half of plane. Note that the total action in (3.5) $S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-)$ describes the closed-path motion which is reversible in time.

As in Sec.1 the new time variables

$$T_\pm = T \pm \tau \quad (3.6)$$

will be used. Considering $ImE \rightarrow +0$ we can consider T and τ as the independent quantities:

$$0 \leq T \leq \infty, \quad -\infty \leq \tau \leq \infty. \quad (3.7)$$

Under this prescriptions the boundary condition $x_+(T_+) = x_-(T_-)$ (see (3.5)) has simple form:

$$x_+(T) = x_-(T). \quad (3.8)$$

Now we will introduce mean trajectory $x(t) = (x_+(t) + x_-(t))/2$ and the deviation $e(t)$ from $x(t)$:

$$x_\pm(t) = x(t) \pm e(t). \quad (3.9)$$

Taking into account (3.8) we will have the boundary conditions only for $e(t)$:

$$e(0) = e(T) = 0. \quad (3.10)$$

Introducing new variables we consider $e(t)$ and τ as the fluctuating, virtual, quantities. We will calculate the corresponding integrals over e and τ perturbatively. In zero order over e and τ , i.e. in the quasiclassical approximation, x is the classical path and T is the total time of classical motion. Note that one can do surely the linear transformations (3.9) in the path integrals.

Let us now extract the linear over e and τ terms from the closed-path action:

$$S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-) = -2\tau H_T(x) - \int_{C^{(+)}(T)} dt e(\ddot{x} + v'(x)) - \tilde{H}_T(x; \tau) - V_T(x, e), \quad (3.11)$$

where

$$C^{(+)}(T) = C_+(T) + C_-(T) \quad (3.12)$$

is the total-time path, H_T is the Hamiltonian:

$$2H_T(x) = -\frac{\partial}{\partial T}(S_{C_+(T)}(x) - S_{C_-(T)}(x)), \quad (3.13)$$

and

$$-\tilde{H}_T(x; \tau) = S_{C_+(T+\tau)}(x) - S_{C_-(T-\tau)}(x) + 2\tau H_T(x), \quad (3.14)$$

$$-V_T(x, e) = S_{C_+(T)}(x + e) - S_{C_-(T)}(x - e) + \int_{C^{(+)}(T)} dt e(\ddot{x} + v'(x)) \quad (3.15)$$

are the remainder terms, and $v'(x) = \partial v(x)/\partial x$. Deriving the decomposition (3.11) the definition

$$C_-(T) = C_+^*(T) \quad (3.16)$$

and the boundary conditions (3.10) was used.

One can find the compact form of $\exp\{-i\tilde{H}_T(x; \tau) - iV_T(x, e)\}$ expansion over τ and e using formulae (2.5):

$$\begin{aligned} \exp\{-i\tilde{H}_T(x; \tau) - iV_T(x, e)\} &= \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau}' - i \int_{C^{(+)}(T)} dt \hat{j}(t)\hat{e}'(t)\right\} \times \\ &\times \exp\{2i\omega\tau + i \int_{C^{(+)}(T)} dt j(t)e(t)\} \exp\{-i\tilde{H}_T(x; \tau') - iV_T(x, e')\}. \end{aligned} \quad (3.17)$$

At the end of calculations the auxiliary variables (ω, τ', j, e') should be taken equal to zero.

Using (3.11) and (3.17) we find from (3.5) that

$$\begin{aligned} R(E) &= 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \hat{j}(t)\hat{e}(t)\right\} \times \\ &\times \int Dx \exp\{-i\tilde{H}_T(x; \tau) - iV_T(x, e)\} \delta(E + \omega - H_T(x)) \prod_t \delta(\ddot{x} + v'(x) - j). \end{aligned} \quad (3.18)$$

The expansion over the differential operators:

$$\begin{aligned} &\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \hat{j}(t)\hat{e}(t) = \\ &= \frac{1}{2i} \frac{\partial}{\partial \omega} \frac{\partial}{\partial \tau} - i \left(\int_{C_+} + \int_{C_-} \right) dt \frac{\delta}{\delta j(t)} \frac{\delta}{\delta e(t)} = \\ &= \frac{1}{2i} \left(\frac{\partial}{\partial \omega} \frac{\partial}{\partial \tau} + Re \int_{C_+} dt \frac{\delta}{\delta j(t)} \frac{\delta}{\delta e(t)} \right) \end{aligned} \quad (3.19)$$

will generate the perturbation series. We propose that they are sumable in Borel sense.

The first δ -function in (3.18) fixes the conservation of energy, see also (1.10):

$$E + \omega = H_T(x) \quad (3.20)$$

where E is the observed energy, $H_T(x)$ is the energy at the mean trajectory at the time moment T and ω is the virtual (fluctuating) energy. The second δ -function

$$\begin{aligned} \prod_t \delta(\ddot{x} + v'(x) - j) &= (2\pi)^2 \int \prod_t \frac{de(t)}{\pi} \delta(e(0)) \delta(e(T)) \times \\ \times e^{-2i\text{Re} \int_{C_+} dt e(\ddot{x} + v'(x) - j)} &= \prod_{t \in C_+(T)} \delta(\text{Re}(\ddot{x} + v'(x) - j)) \delta(\text{Im}(\ddot{x} + v'(x) - j)) \end{aligned} \quad (3.21)$$

fixes the trajectory $x(t)$.

The physical meaning of δ -function (3.21) is as follows [11]. We can consider $(\ddot{x} + v'(x) - j)$ as the total force and $e(t)$ as the virtual deviation from true trajectory $x(t)$. In classical mechanics the virtual work must be equal to zero:

$$(\ddot{x} + v'(x) - j)e(t) = 0, \quad (3.22)$$

since the time-reversible motion is postulated. From this evident dynamical principle one can find the classical equation of motion:

$$\ddot{x} + v'(x) - j = 0, \quad (3.23)$$

since $e(t)$ is arbitrary.

In quantum theories the virtual work is not equal to zero even if the motion is reversible in time by definition (as in our case). But integration over $e(t)$, with boundary conditions (ref36) (see also Sec.6) leads to the same result, see (3.18). So, in quantum theories the unitarity condition play the same role as the d'Alembert's variational principle in classical mechanics.

In (3.23) $j(t)$ describes the external quantum force. The solution $x_j(t)$ of this equation we will find expanding over $j(t)$:

$$x_j(t) = x_c(t) + \int dt_1 G(t, t_1) j(t_1) + \dots \quad (3.24)$$

This is sufficient since $j(t)$ is the auxiliary variable. In this decomposition $x_c(t)$ is the strict solution of unperturbate equation:

$$\ddot{x} + v'(x) = 0 \quad (3.25)$$

Note that the functional δ -function in (3.18) does not contain the end-point values of time $t = 0$ and $t = T$. This means that the initial conditions to the eq.(3.25) are not fixed and the integration over them is assumed since of definition of R , see (1.4).

Inserting (3.24) into (3.23) we find the equation for Green function:

$$(\partial^2 + v''(x_c))_t G(t, t') = \delta(t - t'). \quad (3.26)$$

It is too hard to find the exact solution of this equation since $x_c(t)$ is nontrivial function of t . We will see that the canonical transformation to the conserved quantities can help to avoid this problem, see following section.

4 Canonical transformation

Let us consider motion in the phase space. For this purpose we will insert in (3.18)

$$1 = \int Dp \prod_t \delta(p - \dot{x}). \quad (4.1)$$

In result,

$$\begin{aligned} R(E) = & 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \hat{j}(t)\hat{e}(t)\right\} \times \\ & \times \int Dx Dp \exp\{-i\tilde{H}_T(x; \tau) - iV_T(x, e)\} \times \\ & \times \delta(E + \omega - H_T(x)) \prod_t \delta(\dot{x} - \frac{\partial H_j}{\partial p}) \delta(\dot{p} + \frac{\partial H_j}{\partial x}), \end{aligned} \quad (4.2)$$

where

$$H_j = \frac{1}{2}p^2 + v(x) - jx \quad (4.3)$$

is the total Hamiltonian which is time dependent through $j(t)$.

Instead of pare (x, p) we introduce new pare (θ, h) inserting in (4.2)

$$1 = \int D\theta Dh \prod_t \delta(h - \frac{1}{2}p^2 - v(x)) \delta(\theta - \int^x dx (2(h - v(x)))^{-1/2}). \quad (4.4)$$

Note that the differential measures in (4.2) and (4.4) are δ -like. This allows to change the order of integration and firstly integrate over (x, p) . Since considered transformation is canonical, $\{h, \theta\} = 1$, we find

$$\begin{aligned} R(E) = & 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \hat{j}(t)\hat{e}(t)\right\} \times \\ & \times \int Dh D\theta \exp\{-i\tilde{H}_T(x_c; \tau) - iV_T(x_c, e)\} \times \\ & \times \delta(E + \omega - h(T)) \prod_t \delta(\dot{\theta} - \frac{\partial H_c}{\partial h}) \delta(\dot{h} + \frac{\partial H_c}{\partial \theta}), \end{aligned} \quad (4.5)$$

where

$$H_c = h - jx_c(h, \theta) \quad (4.6)$$

is the transformed Hamiltonian and $x_c(\theta, h)$ is the solution of eq.(3.25) in terms of h and θ .

So, instead of eq.(3.23) we must solve the equations:

$$\dot{h} = j \frac{\partial x_c}{\partial \theta}, \quad \dot{\theta} = 1 - j \frac{\partial x_c}{\partial h}, \quad (4.7)$$

which have a simple structure. Note, that $\partial x_c / \partial \theta$ and $\partial x_c / \partial h$ in the right hand side can be considered as the sources. Expanding the solutions over j we will find the infinite

set of recursive equations. We will use this property of eqs.(4.8) and introduce in the perturbation theory new sources:

$$j_h = j \frac{\partial x_c}{\partial \theta}, \quad j_\theta = j \frac{\partial x_c}{\partial h}. \quad (4.8)$$

For this purpose we will use transformation (2.18):

$$\prod_t \delta(\dot{h} - j \frac{\partial x_c}{\partial \theta}) = e^{-i \int_{C(+)} dt \hat{j}_h(t) \hat{e}_h(t)} e^{i \int_{C(+)} e_h j \frac{\partial x_c}{\partial \theta}} \prod_t \delta(\dot{h} - j_h) \quad (4.9)$$

and

$$\prod_t \delta(\dot{\theta} - 1 + j \frac{\partial x_c}{\partial h}) = e^{-i \int_{C(+)} dt \hat{j}_\theta(t) \hat{e}_\theta(t)} e^{-i \int_{C(+)} e_\theta j \frac{\partial x_c}{\partial h}} \prod_t \delta(\dot{\theta} - 1 - j_\theta). \quad (4.10)$$

The rescaling of source j lead to the rescaling of auxiliary field e . In the new perturbation theory we will have two sources j_h , j_θ and two auxiliary fields e_h , e_θ . Inserting (4.9), (4.10) into (4.5) we find:

$$\begin{aligned} R(E) = 2\pi \int_0^\infty dT \exp\{ \frac{1}{2i} \hat{\omega} \hat{\tau} - i \int_{C(+)} dt (\hat{j}_h(t) \hat{e}_h(t) + \hat{j}_\theta(t) \hat{e}_\theta(t)) \} \times \\ \times \int Dh D\theta \exp\{ -i \tilde{H}_T(x_c; \tau) - i V_T(x_c, e_c) \} \times \\ \times \delta(E + \omega - h(T)) \prod_t \delta(\dot{\theta} - 1 - j_\theta) \delta(\dot{h} - j_h), \end{aligned} \quad (4.11)$$

where

$$e_c = e_h \frac{\partial x_c}{\partial \theta} - e_\theta \frac{\partial x_c}{\partial h} \quad (4.12)$$

carry the symplectic structure of Hamilton's equations of motion.

Hiding the $x_c(t)$ dependence in e_c we solve the problem of the functional determinants, see (4.11), and simplify the equation of motion as much as possible:

$$\dot{h}(t) = j_h(t), \quad \dot{\theta} = 1 + j_\theta(t) \quad (4.13)$$

We will use the boundary conditions

$$h(0) = h_0, \quad \theta(0) = \theta_0, \quad (4.14)$$

as the extension of boundary conditions in (3.5). This lead to the following Green function of transformed perturbation theory:

$$g(t - t') = \Theta(t - t'), \quad (4.15)$$

with the properties of projection operator:

$$\begin{aligned} \int dt dt' g^2(t - t') &= \int dt dt' g(t - t'), \\ \int dt dt' g(t - t') g(t' - t) &= 0 \end{aligned} \quad (4.16)$$

and, at the same time,

$$g(0) = 1/2. \quad (4.17)$$

It is important to note that $Img(t)$ is regular on the real time axis. This is very simplification of the perturbation theory since it eliminate the doubling of degrees of freedom. Here one may use that the analitical continuation to the real time axis and the action of the perturbation-generating operator exponent are commuting operations (see also Sec.2 and the definition (3.19).

In result, shifting C_+ and C_- contours on the real time axis we find:

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}(\hat{\omega}\hat{\tau} + \int_0^T dt_1 dt_2 \Theta(t_1 - t_2)(\hat{e}_h(t_1)\hat{h}(t_2) + \hat{e}_\theta(t_1)\hat{\theta}(t_2)))\right\} \times \\ \times \int dh_0 d\theta_0 \exp\{-i\tilde{H}_T(x_c; \tau) - iV_T(x_c, e_c)\} \delta(E + \omega - h_0 + h(T)), \quad (4.18)$$

where the solutions of eqs.(4.13) was used. In this expression $x_c(t) = x_c(h_0 - h(T), t + \theta_0 - \theta(t))$ and $(h(t), e_h(t), \theta(t), e_\theta(t))$ are the auxiliary fields.

Our perturbation theory describes fluctuations of the initial data (h_0, θ_0) of classical trajectory x_c . The integral form of our perturbation theory is more complicated than of the usual one, over a constant of interaction [13], since demands the investigation of analytic properties of $4N$ -dimensional integrals, where $2N$ is the phase space dimension.

Here we have the new phenomena of reduction of quantum fluctuation, obviously formulated in terms of the plane waves propagation to describe the quantum deformations of the classical trajectory, to the “direct” trajectory fluctuations. The direct deformations were accumulated in the fluctuations of the classical trajectory parameters only, i.e. in the fluctuations of the invariant hypersurface in the phase space on which the trajectory is defined. This statement based on the conservation of total probability.

Taking into account the point of view of Faddeev-Takhtajan [15], that the transformation to the initial data of the inverse scattering problem is the equivalent of the canonical transformation to the action-angle variables, this effect should have the interesting extension to the field theory also.

5 Coordinate transformation

In this section the coordinate transformation of two dimensional model with potential

$$v = v((x_1^2 + x_2^2)^{1/2}) \quad (5.1)$$

will be considered. Repeating calculations of previous sections,

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \hat{j}(t) \hat{e}(t)\right\} \times \\ \times \int D^{(2)} M(x) \exp\{-i\tilde{H}_T(x; \tau) - iV_T(x, e)\}, \quad (5.2)$$

where the δ -like Dirak's differential measure

$$D^{(2)}M(x) = \delta(E + \omega - H_T(x)) \prod_t \delta^{(2)}(\ddot{x} + v'(x) - j) d^2x(t). \quad (5.3)$$

In the classical mechanics the problem with potential (5.1) is solved in the cylindrical coordinates:

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi. \quad (5.4)$$

We insert in (5.2)

$$1 = \int Dr D\phi \prod_t \delta(r - (x_1^2 + x_2^2)^{1/2}) \delta(\phi - \arctan \frac{x_2}{x_1}). \quad (5.5)$$

to perform the transformation. Note that the transformation (5.4) is not canonical. In result we will find a new measure:

$$D^{(2)}M(r, \phi) = \delta(E + \omega - H_T(x)) \prod_t dr d\phi J(r, \phi), \quad (5.6)$$

where the Jacobian of transformation

$$J(r, \phi) = \int \prod d^2x \delta^{(2)}(\ddot{x} + v'(x) - j) \delta(\phi - \arctan \frac{x_2}{x_1}) \delta(r - (x_1^2 + x_2^2)^{1/2}) \quad (5.7)$$

is the product of two δ -functions:

$$J(r, \phi) = \prod_t r^2(t) \delta(\ddot{r} - \dot{\phi}^2 r + v'(r) - j_r) \delta(\partial_t(\dot{\phi} r^2) - r j_\phi), \quad (5.8)$$

where $v'(r) = \partial v(r)/\partial r$ and

$$j_r = j_1 \cos \phi + j_2 \sin \phi, \quad j_\phi = -j_1 \sin \phi + j_2 \cos \phi \quad (5.9)$$

are the components of \vec{j} in the cylindrical coordinates.

It is useful to organize the perturbation theory in terms of j_r and j_ϕ . For this purpose the following transformation of arguments of δ -functions will be used:

$$\prod_t \delta(\ddot{r} - \dot{\phi}^2 r + v'(r) - j_r) = e^{-i \int_{C(+)} dt \hat{j}_r' \hat{e}_r} e^{i \int_{C(+)} dt j_r e_r} \prod_t \delta(\ddot{r} - \dot{\phi}^2 r + v'(r) - j_r') \quad (5.10)$$

and

$$\prod_t \delta(\partial_t(\dot{\phi} r^2) - r j_\phi) = e^{-i \int_{C(+)} dt \hat{j}_\phi' \hat{e}_\phi} e^{i \int_{C(+)} dt j_\phi r e_\phi} \prod_t r(t) \delta(\partial_t(\dot{\phi} r^2) - j_\phi'). \quad (5.11)$$

Here j_r and j_ϕ was defined in (5.9). In result, we get

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i} \hat{\omega} \hat{\tau} - i \int_{C^{(+)}(T)} dt (\hat{j}_r(t) \hat{e}_r(t) + \hat{j}_\phi(t) \hat{e}_\phi(t))\right\} \times \\ \times \int D^{(2)}M(r, \phi) \exp\{-i \tilde{H}_T(x; \tau) - i V_T(x, e_C)\}, \quad (5.12)$$

where

$$D^{(2)}M(r, \phi) = \delta(E + \omega - H_T(r, \phi)) \prod_t r^2(t) dr(t) d\phi(t) \times \\ \times \delta(\ddot{r} - \dot{\phi}^2 r + v'(r) - j_r) \delta(\partial_t(\dot{\phi} r^2) - j_\phi) \quad (5.13)$$

and

$$e_{C,1} = e_r \cos \phi - r e_\phi \sin \phi, \quad e_{C,2} = e_r \sin \phi + r e_\phi \cos \phi. \quad (5.14)$$

This is the final result. The transformation looks quite classically but (5.12) can not be deduced from naive coordinate transformation of initial path integral for amplitude.

Inserting

$$1 = \int Dp Dl \prod_t \delta(p - \dot{r}) \delta(l - \dot{\phi} r^2) \quad (5.15)$$

in (5.12) we can introduce the motion in the phase space with Hamiltonian

$$H_j = \frac{1}{2} p^2 + \frac{l^2}{2r^2} + v(r) - j_r r - j_\phi \phi. \quad (5.16)$$

The Dirak's measure becomes four dimensional:

$$D^{(4)}M(r, \phi, p, l) = \delta(E + \omega - H_T(r, \phi, p, l)) \prod_t dr(t) d\phi(t) dp(t) dl(t) \times \\ \times \delta(\dot{r} - \frac{\partial H_j}{\partial p}) \delta(\dot{\phi} - \frac{\partial H_j}{\partial l}) \delta(\dot{p} + \frac{\partial H_j}{\partial r}) \delta(\dot{l} + \frac{\partial H_j}{\partial \phi}) \quad (5.17)$$

Note absence of the coefficient r^2 in this expression. This is the result of special choice of transformation (5.11).

Since the Hamilton's group manifolds are more rich then Lagrange ones the measure (5.17) can be considered as the starting point of farther transformations. One must to note that the (*action, angle*) variables are mostly useful [15]. Note also that to avoid the technical problems with equations of motion and with functional determinants it is useful to linearise the equation of motion hiding nonlinear terms in the corresponding auxiliary fields.

6 Concluding remarks

One can note that our description looks like the description of classical system under the influence of random *external* force. This gives the possibility to perform classically the transformations of the measure. It allows also to use high resources of classical mechanics.

As the example let us consider the H -atom problem. The hidden (dynamical) symmetry [1] means that the unperturbate Hamiltonian of the problem $h = h(I_1 + I_2)$, where I_k are the action variables. This allows to map the problem on the $O(4)$ sphere in the (action, angle) phase space (this transformation is not canonical: the canonical transformation maps the problem on the $O(2) \times O(2)$ torus) and following Dowker's theorem the

problem on $O(4)$ sphere must be exactly quasiclassical. But previous experience shows that each transformation generates the corresponding sources of quantum excitations. We will have the same for H -atom problem.

The solution of this problem is as follows. The expansion of the operator exponents, which generates the perturbation series, gives the identical to zero contribution, living only first, quasiclassical, term. The reason of this cancellations is the special symplectic structure of the perturbation-generating operator exponent on $O(4)$ sphere, i.e. of the hidden symmetry. This solution of H -atom problem will be published.

The interesting possibility may arise also in connection with Kolmogorov- Arnold-Moser (KAM) theorem [15]: the system which is not strictly integrable can show the stable motion peculiar to integrable systems. This is the argument in favor of the idea that there may be another mechanism of suppression of the quantum excitations.

One can note that the transformed perturbation theory describes only the retarded quantum fluctuations since $\partial x_c/\partial h_0$ and $\partial x_c/\partial \phi_0$ was considered as the sources, see also the definition of Green function (4.15). This feature of the theory can lead to the time irreversibility of quantum processes and must be explained.

The starting expression (3.5) describes the reversible in time motion since total action $S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-)$ is time reversible. But the unitarity condition forced us to consider the interference picture between expanding (with phase $\exp\{iS_{C_+(T_+)}(x_+)\}$) and converging ($\exp\{-iS_{C_-(T_-)}(x_-)\}$) waves. This is fixed by the boundary conditions: $x_+(0) = x_-(0)$ and $x_+(T_+) = x_-(T_-)$. Despite this fact the quantum theory remain time reversible up to canonical transformation to the invariant hypersurface of the constant energy. The causal Green function $G(t - t')$, see (3.26), describes as advanced as well retarded perturbations and the theory contains the doubling of degrees of freedom. It means that the theory “keep in mind” the time reversibility.

But after the canonical transformation, using above mentioned boundary conditions, and continuing the theory to the real times, the memory of doubling of the degrees of freedom disappears and the theory becomes time irreversible.

The key step in this calculations was an extraction of the classical trajectory x_c which can not be defined without definition of boundary conditions. Just this fact introduces the time vector and the quantum perturbations (of the transformed perturbation theory) follow this vector (in the nontransformed perturbation theory this effect is hidden in the impossibility to find the strict solution of eq.(3.26)). Therefore, the considered irreversibility of an quantum-mechanical motion have most probably the imaginary character since the symmetric, one particle, boundary conditions are required.

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